

# A NOTE ON THE VALIDITY OF BOGOLIUBOV CORRECTION TO MEAN-FIELD DYNAMICS

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**ABSTRACT.** We study the norm approximation to the Schrödinger dynamics of  $N$  bosons in  $\mathbb{R}^3$  with an interaction potential of the form  $N^{3\beta-1}w(N^\beta(x-y))$ . Assuming that in the initial state the particles outside of the condensate form a quasi-free state with finite kinetic energy, we show that in the large  $N$  limit, the fluctuations around the condensate can be effectively described using Bogoliubov approximation for all  $0 \leq \beta < 1/2$ . The range of  $\beta$  is expected to be optimal for this large class of initial states.

## 1. INTRODUCTION

We are interested in the norm approximation of the Schrödinger evolution

$$\Psi_N(t) = e^{-itH_N} \Psi_N(0) \quad (1)$$

on the bosonic Hilbert space  $\mathfrak{H}^N = \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^3)$ . Here

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N-1} \sum_{1 \leq j < k \leq N} w_N(x_j - x_k)$$

is the Hamiltonian of a system of  $N$  identical bosons in  $\mathbb{R}^3$ . The interaction potential is taken of the delta-type form

$$w_N(x-y) = N^{3\beta} w(N^\beta(x-y)).$$

The parameter  $\beta \geq 0$  is fixed and  $w \in C_0^1(\mathbb{R}^3)$  is non-negative and spherically symmetric decreasing.

From the physical point of view, the initial state  $\Psi_N(0)$  may be interpreted as a ground state of a trapped system and the time evolution  $\Psi_N(t)$  in (1) is observed when the trapping potential is turned off. Thus, motivated by the results on ground states in [16] (see also [22, 8, 4, 20]), we expect that

$$\Psi_N(0) \approx \sum_{n=0}^N u(0)^{\otimes(N-n)} \otimes_s \psi_n(0) \quad (2)$$

in norm for  $N$  large. Here  $u(0)$  is a normalized function in  $L^2(\mathbb{R}^3)$  describing the Bose-Einstein condensate and  $(\psi_n(0))_{n=0}^\infty$  is a quasi-free state describing the fluctuations around the condensate.

We will show that if  $\Psi_N(0)$  satisfies (2), then for every  $t > 0$ , we have

$$\Psi_N(t) \approx \sum_{n=0}^N u(t)^{\otimes(N-n)} \otimes_s \psi_n(t) \quad (3)$$

in norm for  $N$  large. Here  $u(t) \in L^2(\mathbb{R}^3)$  is determined by a mean-field (Hartree) equation and  $(\psi_n(t))_{n=0}^\infty$  is a quasi-free state governed by a quadratic (Bogoliubov) Hamiltonian on Fock space.

The approximation (3) has been first established in [15] for  $\beta = 0$ , and then extended to  $0 \leq \beta < 1/3$  in [18]. This range of  $\beta$  seems to be optimal if we only assume that  $(\psi_n(0))_{n=0}^\infty$  in (2) is a quasi-free state. In the present work, we will make an additional assumption (still physically reasonable) that  $(\psi_n(0))_{n=0}^\infty$  has finite kinetic energy, and prove (3) for all  $0 \leq \beta < 1/2$ . Note that when  $\beta > 1/3$ , the range of the interaction potential is much smaller than the average distance between the particles, and hence every particle essentially interacts only with itself. This so-called self-interaction regime is physically more relevant and mathematically more challenging than the mean-field regime  $\beta < 1/3$ .

An analogue of (3) related to the fluctuations around coherent states in Fock space has been justified in [13, 6, 7, 11, 12] for  $\beta = 0$ , in [9] for  $\beta < 1/3$  and in [14] for  $\beta < 1/2$ . In particular, our result is comparable to [14], but our method is different and it can be used to simplify the proof in [14]. Thanks to a heuristic argument in [14], we also expect that the range  $0 \leq \beta < 1/2$  is optimal for the approximation (3) to hold, as soon as  $u(t)$  is still decoupled from the equation for  $(\psi_n(t))_{n=0}^\infty$ .

When  $\beta > 1/2$ , the effective equations for  $u(t)$  and  $(\psi_n(t))_{n=0}^\infty$  in (3) have to be modified to take two-body scattering processes into account. This step has been carried out in the Fock space setting in [2, 10], but it is still open in the  $N$ -particle setting.

Note that the norm convergence (3) is much more precise than the usual convergence of density matrices in the context of the Bose-Einstein condensation. In particular, our result can be interpreted as a second order correction to the leading order result in [5]. We refer to [18] for a further discussion and an extended list of literature in this direction.

The precise statement of our result is given in the next section.

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## 2. MAIN RESULT

In our paper, the condensate is governed by the Hartree equation

$$\begin{cases} i\partial_t u(t) = (-\Delta + w_N * |u(t)|^2 - \mu_N(t))u(t), \\ u(t=0) = u(0). \end{cases} \quad (4)$$

Here we can choose the phase

$$\mu_N(t) = \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |u(t, x)|^2 w_N(x - y) |u(t, y)|^2 dx dy$$

to ensure an energy compatibility (see [15] for further explanations). The well-posedness of the Hartree equation is recalled in Lemma 2.

To describe the fluctuations around the condensate, it is natural to introduce the Fock space

$$\mathcal{F}(\mathfrak{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{H}^n = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \bigotimes_{\text{sym}}^n \mathfrak{H}, \quad \mathfrak{H} = L^2(\mathbb{R}^3).$$

On this Fock space, we define the creation and annihilation operators  $a^*(f)$ ,  $a(f)$ , with  $f \in \mathfrak{H}$ , by

$$(a^*(f)\Psi)(x_1, \dots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} f(x_j) \Psi(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}),$$

$$(a(f)\Psi)(x_1, \dots, x_n) = \sqrt{n} \int \overline{f(x_n)} \Psi(x_1, \dots, x_n) dx_n, \quad \forall \Psi \in \mathfrak{H}^n, \forall n.$$

These operators satisfy the canonical commutation relations (CCR)

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0, \quad [a(f), a^*(g)] = \langle f, g \rangle, \quad \forall f, g \in \mathfrak{H}.$$

Equivalently, we can define the operator-valued distributions  $a_x^*$  and  $a_x$ , with  $x \in \mathbb{R}^3$ , by

$$a^*(f) = \int_{\mathbb{R}^3} f(x) a_x^* dx, \quad a(f) = \int_{\mathbb{R}^3} \overline{f(x)} a_x dx, \quad \forall f \in \mathfrak{H}.$$

They satisfy

$$[a_x^*, a_y^*] = [a_x, a_y] = 0, \quad [a_x, a_y^*] = \delta(x - y), \quad \forall x, y \in \mathbb{R}^3.$$

These operators allow us to express operators on Fock space in a convenient way. For example, for every operator  $h$  on  $L^2(\mathbb{R}^3)$  with kernel  $h(x, y)$ , we can write

$$d\Gamma(h) := 0 \oplus \bigoplus_{n=0}^{\infty} \sum_{j=1}^n h_j = \int_{\mathbb{R}^3} a_x^* h a_x dx = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} h(x, y) a_x^* a_y dx dy.$$

In particular,  $\mathcal{N} = d\Gamma(1)$  is called the number operator.

In our approximation (2)-(3), the particles outside of the condensate are described by a unit vector  $\Phi(t) = (\psi_n(t))_{n=0}^{\infty}$  in the excited Fock space

$$\mathcal{F}_+(t) = \bigoplus_{n=0}^{\infty} \bigotimes_{\text{sym}}^n \mathfrak{H}_+(t), \quad \mathfrak{H}_+(t) = \{u(t)\}^{\perp} = Q(t)\mathfrak{H}, \quad Q(t) := 1 - |u(t)\rangle\langle u(t)|.$$

This vector is governed by the Bogoliubov equation

$$\begin{cases} i\partial_t \Phi(t) = \mathbb{H}(t)\Phi(t), \\ \Phi(t=0) = \Phi(0), \end{cases} \quad (5)$$

where

$$\mathbb{H}(t) := d\Gamma(h(t)) + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left( K_2(t, x, y) a_x^* a_y^* + \overline{K_2(t, x, y)} a_x a_y \right) dx dy,$$

$$h(t) = -\Delta + |u(t, \cdot)|^2 * w_N - \mu_N(t) + Q(t) \tilde{K}_1(t) Q(t),$$

$$K_2(t, \cdot, \cdot) = Q(t) \otimes Q(t) \tilde{K}_2(t, \cdot, \cdot).$$

Here  $\tilde{K}_1(t)$  is the operator on  $\mathfrak{H}$  with kernel  $\tilde{K}_1(t, x, y) = u(t, x)w_N(x - y)\overline{u(t, y)}$ , and  $\tilde{K}_2(t, x, y) = u(t, x)w_N(x - y)u(t, y)$ . A heuristic derivation of (5) will be revised in Section 4.

We will restrict our attention to quasi-free states. Recall that a unit vector  $\Psi \in \mathcal{F}(\mathfrak{H})$  is called a quasi-free state if it has finite particle number expectation, namely  $\langle \Psi, \mathcal{N}\Psi \rangle < \infty$ , and satisfies Wick's Theorem:

$$\begin{aligned} \langle \Psi, a^\#(f_1)a^\#(f_2)\cdots a^\#(f_{2n-1})\Psi \rangle &= 0, \\ \langle \Psi, a^\#(f_1)a^\#(f_2)\cdots a^\#(f_{2n})\Psi \rangle &= \sum_{\sigma} \prod_{j=1}^n \langle \Psi, a^\#(f_{\sigma(2j-1)})a^\#(f_{\sigma(2j)})\Psi \rangle \end{aligned}$$

for all  $f_1, \dots, f_n \in \mathfrak{H}$  and for all  $n$ . Here  $a^\#$  is either the creation or annihilation operator and the sum is taken over all permutations  $\sigma$  satisfying  $\sigma(2j-1) < \min\{\sigma(2j), \sigma(2j+1)\}$  for all  $j$ . By the definition, any quasi-free state is determined uniquely (up to a phase) by its one-body density matrices  $\gamma_\Psi : \mathfrak{H} \rightarrow \mathfrak{H}$  and  $\alpha_\Psi : \overline{\mathfrak{H}} \equiv \mathfrak{H}^* \rightarrow \mathfrak{H}$  which are defined by

$$\langle f, \gamma_\Psi g \rangle = \langle \Psi, a^*(g)a(f)\Psi \rangle, \quad \langle f, \alpha_\Psi \bar{g} \rangle = \langle \Psi, a(g)a(f)\Psi \rangle, \quad \forall f, g \in \mathfrak{H}.$$

In [18], we proved that if  $\Phi(0)$  is a quasi-free state, then the solution  $\Phi(t)$  to (5) is a quasi-free state for all  $t > 0$  and  $(\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$  is the unique solution to the system

$$\begin{cases} i\partial_t \gamma = h\gamma - \gamma h + K_2 \alpha - \alpha^* K_2^*, \\ i\partial_t \alpha = h\alpha + \alpha h^T + K_2 + K_2 \gamma^T + \gamma K_2, \\ \gamma(t=0) = \gamma_{\Phi(0)}, \quad \alpha(t=0) = \alpha_{\Phi(0)}. \end{cases} \quad (6)$$

Here  $K_2$  is interpreted as an operator  $\mathfrak{H}^* \rightarrow \mathfrak{H}$  with kernel  $K_2(t, x, y)$ . Note that (6) is similar (but not identical) to the equations studied in [9, 14, 1]. The well-posedness of (5)-(6) is recalled in Lemma 3.

Now we are ready to state our main result.

**Theorem 1** (Validity of Bogoliubov dynamics). *Let  $0 \leq \beta < 1/2$ .*

- *Let  $u(t)$  satisfy the Hartree equation (4), where the (possibly  $N$ -dependent) initial state  $u(0, \cdot)$  satisfies*

$$\|u(0, \cdot)\|_{W^{\ell,1}(\mathbb{R}^3)} \leq \kappa_0$$

*for  $\ell$  sufficiently large and for a constant  $\kappa_0 > 0$  independent of  $N$ .*

- *Let  $\Phi(t) = (\psi_n(t))_{n=0}^\infty \in \mathcal{F}_+(t)$  satisfy the Bogoliubov equation (5), where the (possibly  $N$ -dependent) initial state  $\Phi(0)$  is a quasi-free state in  $\mathcal{F}_+(0)$  satisfying*

$$\langle \Phi(0), \mathcal{N}\Phi(0) \rangle \leq \kappa_\varepsilon N^\varepsilon \quad \text{and} \quad \langle \Phi(0), d\Gamma(1 - \Delta)\Phi(0) \rangle \leq \kappa_\varepsilon N^{\beta+\varepsilon} \quad (7)$$

*for all  $\varepsilon > 0$ , where the constant  $\kappa_\varepsilon > 0$  is independent of  $N$ .*

- *Let  $\Psi_N(t)$  satisfy the Schrödinger equation (1) with the initial state*

$$\Psi_N(0) = \sum_{n=0}^N u(0)^{\otimes(N-n)} \otimes_s \psi_n(0) = \sum_{n=0}^N \frac{(a^*(u(0)))^{N-n}}{\sqrt{(N-n)!}} \psi_n(0). \quad (8)$$

Then for all  $\varepsilon > 0$  and for all  $t > 0$  we have

$$\left\| \Psi_N(t) - \sum_{n=0}^N u(t)^{\otimes(N-n)} \otimes_s \psi_n(t) \right\|_{\mathfrak{H}^N}^2 \leq C_\varepsilon (1+t)^{1+\varepsilon} N^{(2\beta-1+\varepsilon)/2} \quad (9)$$

where the constant  $C_\varepsilon > 0$  depends only on  $\kappa_0$  and  $\varepsilon$ .

Strictly speaking, the initial state  $\Psi_N(0)$  is not normalized. However, its norm converges to 1 very fast when  $N \rightarrow \infty$  (we will see it from the proof). We ignore this trivial normalization in the statement of Theorem 1 for simplicity.

Since  $\Psi_N(0)$  is expected to be the ground state of a trapped system with the interaction potential  $w_N(x-y)$ , the initial data  $u(0, \cdot)$  and  $\Phi(0)$  are allowed to depend on  $N$ . In particular, the assumptions (7) on  $\Phi(0)$  are motivated by the ground state properties of quadratic Hamiltonians (see Remark 8). More generally, we can also assume that (7) holds for *some*  $\varepsilon > 0$ , and replace the right side of (9) by  $C_\varepsilon (1+t)^{1+\varepsilon} N^{(2\beta-1+9\varepsilon)/2}$  (see the estimate (69) in the proof).

Our proof builds on ideas in [15, 18], where the case  $0 \leq \beta < 1/3$  was studied. However, the extension to  $\beta < 1/2$  requires several new tools, most notably a new kinetic estimate for the particles outside of the condensate (see Lemma 10). Our method can be applied to study the norm approximation in Fock space, for example to simplify significantly the proof in [14]. The range  $0 \leq \beta < 1/2$  is expected to be optimal under the assumptions on the initial states in Theorem 1.

The paper is organized as follows. We will revise the well-posedness of the Hartree equation (4) and the Bogoliubov equation (5) in Section 3. In section 4, we reformulate the problem using a unitary transformation from  $\mathfrak{H}^N$  to a truncated Fock space, following ideas in [16, 15]. Then we provide several estimates which are useful to implement Bogoliubov's approximation. The proof of Theorem 1 is presented in Section 5.

### 3. WELL-POSEDNESS OF THE EFFECTIVE EQUATIONS

From [9, Prop. 3.3 & Cor. 3.4] we have the following well-posedness of the Hartree equation.

**Lemma 2.** *If  $u(0, \cdot) \in H^2(\mathbb{R}^3)$ , then the Hartree equation (4) has a unique global solution  $u \in C([0, \infty), H^2(\mathbb{R}^3)) \cap C^1((0, \infty), L^2(\mathbb{R}^3))$ . Moreover, if  $u(0, \cdot) \in W^{\ell,1}(\mathbb{R}^3)$  with  $\ell$  sufficiently large, then  $\|u(t, \cdot)\|_{H^2} \leq C$ ,  $\|\partial_t u(t, \cdot)\|_{L^2} \leq C$  and*

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} + \|\partial_t u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{(1+t)^{3/2}}$$

for a constant  $C$  depending only on  $\|u(0)\|_{W^{\ell,1}(\mathbb{R}^3)}$ .

From now on, we always assume that  $u(0, \cdot) \in W^{\ell,1}(\mathbb{R}^3)$  with  $\ell$  sufficiently large. We will also denote by  $C$  a general constant depending only on  $\|u(0, \cdot)\|_{W^{\ell,1}}$  (whose value can be changed from line to line). Indeed, more precisely,  $C$  depends only on  $\kappa_0$  in the condition  $\|u(0, \cdot)\|_{W^{\ell,1}(\mathbb{R}^3)} \leq \kappa_0$  (c.f. Theorem 1).

Next, we recall the well-posedness of the Bogoliubov equation from [15, Theorem 7] and [18, Prop. 4].

**Lemma 3.** *For every initial state  $\Phi(0)$  in the quadratic form domain of  $\mathcal{Q}(\mathrm{d}\Gamma(1 - \Delta))$ , the Bogoliubov equation (5) has a unique global solution  $\Phi \in C([0, \infty), \mathcal{F}(\mathfrak{H})) \cap L_{\mathrm{loc}}^\infty((0, \infty), \mathcal{Q}(\mathrm{d}\Gamma(1 - \Delta)))$ . Moreover, if  $\Phi(0)$  is a quasi-free state in  $\mathcal{F}_+(0)$ , then  $\Phi(t)$  is a quasi-free state in  $\mathcal{F}_+(t)$  and*

$$\langle \Phi(t), \mathcal{N}\Phi(t) \rangle \leq C \left( \langle \Phi(0), \mathcal{N}\Phi(0) \rangle^2 + [\log(2 + t)]^2 \right).$$

We have two remarks on the Bogoliubov equation (5). First, although the Bogoliubov Hamiltonian  $\mathbb{H}(t)$  is not necessarily bounded from below and has not been defined as a self-adjoint operator, the solution  $\Phi(t)$  to (5) can be still interpreted as an evolution generated by quadratic forms (see [15, Theorems 7, 8] for further discussion). Second, when  $\Phi(t)$  is a quasi-free state, then the Bogoliubov equation (5) becomes equivalent to the system (6) (see [18, Prop. 4] for more details), but we will not need this fact in the rest of paper.

The main new result of this section is the following kinetic estimate.

**Lemma 4.** *Assume that  $\Phi(0)$  is a quasi-free state in  $\mathcal{F}_+(0)$  satisfying*

$$\langle \Phi(0), \mathrm{d}\Gamma(1 - \Delta)\Phi(0) \rangle \leq \kappa_\varepsilon N^{\beta+\varepsilon}$$

*for some  $\varepsilon > 0$ , where the constant  $\kappa_\varepsilon$  is independent of  $N$ . Then*

$$\langle \Phi(t), \mathrm{d}\Gamma(1 - \Delta)\Phi(t) \rangle \leq C_\varepsilon N^{\beta+\varepsilon}, \quad \forall t > 0.$$

Hereafter,  $C_\varepsilon$  is a general constant depending only on  $\|u(0, \cdot)\|_{W^{\ell,1}}$  (more precisely, on  $\kappa_0$  in the condition  $\|u(0, \cdot)\|_{W^{\ell,1}} \leq \kappa_0$ ) and  $\varepsilon$ .

To prove Lemma 4, we will need a general lower bound on the ground state energy of quadratic Hamiltonians.

**Lemma 5.** *Let  $H > 0$  be a self-adjoint operator on  $\mathfrak{H}$ . Let  $K : \overline{\mathfrak{H}} \equiv \mathfrak{H}^* \rightarrow \mathfrak{H}$  be an operator with kernel  $K(x, y) \in \mathfrak{H}^2$ . Assume that  $KH^{-1}K^* \leq H$  and that  $H^{-1/2}K$  is Hilbert-Schmidt. Then*

$$\mathrm{d}\Gamma(H) + \frac{1}{2} \iint \left( K(x, y) a_x^* a_y^* + \overline{K(x, y)} a_x a_y \right) dx dy \geq -\frac{1}{2} \|H^{-1/2}K\|_{\mathrm{HS}}^2.$$

This result is taken from [19, Lemma 9] (see also [3, Theorem 5.4]). Note that

$$\|H^{-1/2}K\|_{\mathrm{HS}}^2 = \iint |H_x^{-1/2}K(x, y)|^2 dx dy =: \|H_x^{-1/2}K(\cdot, \cdot)\|_{L^2}^2.$$

Here we write  $H_x$  to mention that the operator  $H$  acts on the  $x$ -variable.

If we apply Lemma 5 with  $H = 1 + \|K_2\|$  and  $K = \pm K_2$ , then we get

$$\pm \frac{1}{2} \iint \left( K_2(t, x, y) a_x^* a_y^* + \overline{K_2(t, x, y)} a_x a_y \right) dx dy \leq CN + \frac{CN^{3\beta}}{(1+t)^3}. \quad (10)$$

Here we have used the bound on  $\|K_2\|$  in (20) and

$$\begin{aligned} \|K_2(t, \cdot, \cdot)\|_{L^2}^2 &\leq \|\tilde{K}_2(t, \cdot, \cdot)\|_{L^2}^2 = \iint |u(t, x)|^2 |w_N(x - y)|^2 |u(t, y)|^2 dx dy \\ &\leq \|w_N\|_{L^2}^2 \|u(t, \cdot)\|_{L^\infty}^2 \|u(t, \cdot)\|_{L^2}^2 \leq \frac{CN^{3\beta}}{(1+t)^3} \end{aligned} \quad (11)$$

by Lemma 2. In order to improve the factor  $N^{3\beta}$  in (10), we will apply Lemma 5 with  $H = 1 - \Delta$ . We will need the following estimate.

**Lemma 6.** *For all  $\varepsilon > 0$  we have*

$$\|(1 - \Delta_x)^{-1/2} K_2(t, \cdot, \cdot)\|_{L^2}^2 + \|(1 - \Delta_x)^{-1/2} \partial_t K_2(t, \cdot, \cdot)\|_{L^2}^2 \leq \frac{C_\varepsilon N^{\beta+\varepsilon}}{(1+t)^3}.$$

*Proof.* We will present a detailed proof for  $\partial_t K_2(t)$  and  $K_2(t)$  can be treated by the same way. Recall that

$$K_2(t, \cdot, \cdot) = Q(t) \otimes Q(t) \tilde{K}_2(t, \cdot, \cdot), \quad \tilde{K}_2(t, x, y) = u(t, x) w_N(x - y) u(t, y).$$

Hence,

$$\partial_t K_2(t) = \partial_t Q(t) \otimes Q(t) \tilde{K}_2(t) + Q(t) \otimes \partial_t Q(t) \tilde{K}_2(t) + Q(t) \otimes Q(t) \partial_t \tilde{K}_2(t).$$

Since  $\partial_t Q(t) = -|\partial_t u(t)\rangle\langle u(t)| - |u(t)\rangle\langle \partial_t u(t)|$ , we have

$$\begin{aligned} \|\partial_t Q(t) \otimes Q(t) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} &\leq \|(\partial_t Q(t) \otimes 1) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \\ &\leq \|(|\partial_t u(t)\rangle\langle u(t)| \otimes 1) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} + \|(|u(t)\rangle\langle \partial_t u(t)| \otimes 1) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \end{aligned}$$

Using Lemma 2 and  $\|w_N\|_{L^1} = \|w\|_{L^1}$ , it is straightforward to see that

$$\begin{aligned} &\left\| (|\partial_t u\rangle\langle u| \otimes 1) \tilde{K}_2(t, \cdot, \cdot) \right\|_{L^2}^2 \\ &= \iint \left| \int \overline{u(t, z)} u(t, z) w_N(z - y) u(t, y) dz \right|^2 |\partial_t u(t, x)|^2 dx dy \\ &\leq \|u(t, \cdot)\|_{L^\infty}^4 \|w_N\|_{L^1}^2 \|u(t, \cdot)\|_{L^2}^2 \|\partial_t u(t, \cdot)\|_{L^2}^2 \leq \frac{C}{(1+t)^3}. \end{aligned}$$

combining this with similar estimates, we find that

$$\begin{aligned} &\|\partial_t Q(t) \otimes Q(t) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} + \|Q(t) \otimes \partial_t Q(t) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \\ &\leq \|(\partial_t Q(t) \otimes 1) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} + \|(1 \otimes \partial_t Q(t)) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \leq \frac{C}{(1+t)^{3/2}}. \end{aligned} \tag{12}$$

By the same argument, we also obtain

$$\|(1 - Q(t) \otimes Q(t)) \partial_t \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \leq \frac{C}{(1+t)^{3/2}}.$$

Note that  $(1 - \Delta_x)^{-1/2} \leq 1$  on  $L^2$ , and hence we can insert  $(1 - \Delta_x)^{-1/2}$  into the above  $L^2$  norm estimates for free. It remains to show that

$$\|(1 - \Delta_x)^{-1/2} \partial_t \tilde{K}_2(t, \cdot, \cdot)\|_{L^2}^2 \leq \frac{C_\varepsilon N^{\beta+\varepsilon}}{(1+t)^3}, \quad \forall \varepsilon > 0. \tag{13}$$

Similarly to (11), we have

$$\|\partial_t \tilde{K}_2(t, \cdot, \cdot)\|_{L^2}^2 \leq \frac{CN^{3\beta}}{(1+t)^3}.$$

Therefore, by interpolation (more precisely, by Hölder's inequality in Fourier space), (13) follows from the following estimate

$$\|(1 - \Delta_x)^{-3/4-\varepsilon} \partial_t \tilde{K}_2(t, \cdot, \cdot)\|_{L^2}^2 \leq \frac{C_\varepsilon}{(1+t)^3}, \quad \forall \varepsilon > 0. \tag{14}$$

It suffices to show that

$$\|(1 - \Delta_x)^{-3/4-\varepsilon} f(t, \cdot, \cdot)\|_{L^2}^2 \leq \frac{C_\varepsilon}{(1+t)^3}, \quad \forall \varepsilon > 0 \tag{15}$$

with  $f(t, x, y) = \partial_t u(t, x) w_N(x - y) u(t, y)$ . The bound (15) can be proved using an argument in [9]. Let us compute the Fourier transform:

$$\begin{aligned} \widehat{f}(t, p, q) &= \iint u(t, x) w_N(x - y) (\partial_t u)(t, y) e^{-2\pi i(p \cdot x + q \cdot y)} dx dy \\ &= \iint u(t, y + z) w_N(z) (\partial_t u)(t, y) e^{-2\pi i(p \cdot (y+z) + q \cdot y)} dz dy \\ &= \int w_N(z) (\widehat{u_z \partial_t u})(t, p + q) e^{-2\pi i p \cdot z} dz \end{aligned}$$

where  $u_z(t, \cdot) := u(t, z + \cdot)$ . By the Cauchy-Schwarz inequality,

$$\left| \widehat{f}(t, p, q) \right|^2 \leq \|w_N\|_{L^1} \int |w_N(z)| \cdot |(\widehat{u_z \partial_t u})(t, p + q)|^2 dz.$$

Using Plancherel's Theorem, we can estimate

$$\begin{aligned} \|(1 - \Delta_x)^{-3/4-\varepsilon} f(t, \cdot, \cdot)\|_{L^2}^2 &= \iint (1 + |2\pi p|^2)^{-3/2-2\varepsilon} \left| \widehat{f}(t, p, q) \right|^2 dp dq \\ &\leq \|w_N\|_{L^1} \iint (1 + |2\pi p|^2)^{-3/2-2\varepsilon} |w_N(z)| \cdot |(\widehat{u_z \partial_t u})(t, p + q)|^2 dp dq dz. \end{aligned}$$

By Lemma 2,

$$\begin{aligned} \int |(\widehat{u_z \partial_t u})(t, p + q)|^2 dq &= \|(u_z \partial_t u)(t, \cdot)\|_{L^2}^2 \\ &\leq \|u(t, \cdot)\|_{L^\infty}^2 \|\partial_t u(t, \cdot)\|_{L^2}^2 \leq \frac{C}{(1+t)^3}. \end{aligned}$$

Therefore, (15) follows from  $\|w_N\|_{L^1} = \|w\|_{L^1}$  and the fact that

$$\int (1 + |2\pi p|^2)^{-3/2-2\varepsilon} dp \leq C_\varepsilon < \infty.$$

Thus (15) holds true. By the same argument, we obtain a similar inequality with  $f(t, x, y)$  replaced by  $u(t, x) w_N(x - y) \partial_t u(t, y)$ . Combining these two estimates, we deduce (14). This completes the proof.  $\square$

Now we apply Lemmas 5 and 6 to bound  $\mathbb{H}(t)$ .

**Lemma 7.** *For every  $\varepsilon > 0$  and  $\eta > 0$ , we have*

$$\begin{aligned} \pm(\mathbb{H}(t) + d\Gamma(\Delta)) &\leq \eta d\Gamma(1 - \Delta) + \frac{C_\varepsilon(\mathcal{N} + N^{\beta+\varepsilon})}{\eta(1+t)^3}, \\ \pm \partial_t \mathbb{H}(t) &\leq \eta d\Gamma(1 - \Delta) + \frac{C_\varepsilon(\mathcal{N} + N^{\beta+\varepsilon})}{\eta(1+t)^3}, \\ \pm i[\mathbb{H}(t), \mathcal{N}] &\leq \eta d\Gamma(1 - \Delta) + \frac{C_\varepsilon(\mathcal{N} + N^{\beta+\varepsilon})}{\eta(1+t)^3}, \end{aligned}$$

as quadratic forms on  $\mathcal{F}(\mathfrak{H})$ . The constant  $C_\varepsilon$  is independent of  $\eta$ .

*Proof.* First, we consider

$$\mathbb{H}(t) + d\Gamma(\Delta) = d\Gamma(h + \Delta) + \frac{1}{2} \iint \left( K_2(t, x, y) a_x^* a_y^* + \overline{K_2(t, x, y)} a_x a_y \right) dx dy.$$

Recall that

$$h + \Delta = |u(t, \cdot)|^2 * w_N - \mu_N(t) + Q(t) \widetilde{K}_1(t) Q(t),$$



where  $\tilde{K}_1(t)$  is the operator with kernel  $\tilde{K}_1(t) = u(t, x)w_N(x - y)\overline{u(t, y)}$ . Using Lemma 2, we have

$$\| |u(t, \cdot)|^2 * w_N \|_{L^\infty} \leq \|u(t, \cdot)\|_{L^\infty}^2 \|w_N\|_{L^1} \leq \frac{C}{(1+t)^3}, \quad (16)$$

$$\mu_N(t) = \frac{1}{2} \int |u(t, x)|^2 (|u(t, \cdot)|^2 * w_N)(x) dx \leq \frac{C}{(1+t)^3}. \quad (17)$$

Moreover,  $\|Q(t)\tilde{K}_1(t)Q(t)\| \leq \|\tilde{K}_1(t)\|$  and

$$\begin{aligned} \|\tilde{K}_1(t)\| &= \sup_{\|f\|_{L^2}=1} \left| \iint \overline{f(x)} u(t, x) w_N(x - y) \overline{u(t, y)} f(y) dx dy \right| \\ &\leq \sup_{\|f\|_{L^2}=1} \|u(t, \cdot)\|_{L^\infty}^2 \iint \frac{|f(x)|^2 + |f(y)|^2}{2} w_N(x - y) dx dy \leq \frac{C}{(1+t)^3}. \end{aligned} \quad (18)$$

From (16), (17) and (18) and the triangle inequality, we get

$$\|h + \Delta\| \leq \frac{C}{(1+t)^3}. \quad (19)$$

Similarly to (18), we can bound the operator  $K_2(t) : \mathfrak{H}^* \rightarrow \mathfrak{H}$  as

$$\|K_2(t)\| \leq \frac{C}{(1+t)^3}. \quad (20)$$

Now we apply Lemma 5 with  $H = \eta(1 - \Delta) + \eta^{-1}\|K_2\|^2$  and  $K = \pm K_2$ , where  $\eta > 0$  is arbitrary. Since  $H \geq \|K\|$ , we have

$$H \geq \|K\| \geq K\|K\|^{-1}K^* \geq KH^{-1}K.$$

Moreover, using  $H \geq \eta(1 - \Delta)$  and Lemma 6, we get

$$\|H^{-1/2}K\|_{\text{HS}}^2 \leq \eta^{-1}\|(1 - \Delta)^{-1/2}K_2\|_{\text{HS}}^2 \leq \frac{C_\varepsilon N^{\beta+\varepsilon}}{\eta(1+t)^3}.$$

Therefore, Lemma 5 implies that

$$\begin{aligned} &\pm \frac{1}{2} \iint \left( K_2(t, x, y) a_x^* a_y^* + \overline{K_2(t, x, y)} a_x a_y \right) dx dy \\ &\leq \eta d\Gamma(1 - \Delta) + \eta^{-1}\|K_2\|^2 \mathcal{N} + \frac{C_\varepsilon N^{\beta+\varepsilon}}{\eta(1+t)^3}. \end{aligned} \quad (21)$$

From (19), by using the Cauchy-Schwarz inequality  $C(1+t)^{-3} \leq \eta + C\eta^{-1}(1+t)^{-6}$  we get

$$\pm d\Gamma(h + \Delta) \leq \eta \mathcal{N} + \frac{C\mathcal{N}}{\eta(1+t)^6}$$

for all  $\eta > 0$ . Combining this with (21) (and the obvious bound  $\mathcal{N} \leq d\Gamma(1 - \Delta)$ ) we conclude that

$$\pm \left( \mathbb{H}(t) + d\Gamma(\Delta) \right) \leq \eta d\Gamma(1 - \Delta) + \frac{C_\varepsilon (\mathcal{N} + N^{\beta+\varepsilon})}{\eta(1+t)^3}.$$

The bound on  $\partial_t \mathbb{H}(t)$  is obtained by the same way. Indeed, by Lemma 2,

$$\|\partial_t(|u(t, \cdot)|^2 * w_N)\|_{L^\infty} \leq \|\partial_t |u(t, \cdot)|^2\|_{L^\infty} \|w_N\|_{L^1} \leq \frac{C}{(1+t)^3}, \quad (22)$$

$$|\partial_t \mu_N(t)| \leq 2 \iint |\partial_t(|u(t, x)|^2)| w_N(x - y) |u(t, y)|^2 dx dy \leq \frac{C}{(1+t)^3}, \quad (23)$$

$$\|\partial_t(Q(t)\tilde{K}_1(t)Q(t))\| \leq \|\partial_t\tilde{K}_1(t)\| + 2\|\partial_t Q(t)\| \cdot \|\tilde{K}_1(t)\| \leq \frac{C}{(1+t)^3}. \quad (24)$$

Similarly to (24), we can bound the operator  $\partial_t K_2(t) : \mathfrak{H}^* \rightarrow \mathfrak{H}$  as

$$\|\partial_t K_2(t)\| \leq \frac{C}{(1+t)^3}. \quad (25)$$

Then we apply Lemma 5 with  $H = \eta(1 - \Delta) + \eta^{-1}\|\partial_t K_2\|^2$  and  $K = \pm \partial_t K_2$  and obtain

$$\pm \partial_t \mathbb{H}(t) \leq \eta d\Gamma(1 - \Delta) + \frac{C_\varepsilon(\mathcal{N} + N^{\beta+\varepsilon})}{\eta(1+t)^3}.$$

Finally, since  $[a_x^* a_y^*, \mathcal{N}] = -2a_x^* a_y^*$ , we have

$$\begin{aligned} i[\mathbb{H}(t), \mathcal{N}] &= \frac{i}{2} \iint \left( K_2(t, x, y) [a_x^* a_y^*, \mathcal{N}] + \overline{K_2(t, x, y)} [a_x a_y, \mathcal{N}] \right) dx dy \\ &= - \iint \left( iK_2(t, x, y) a_x^* a_y^* + \overline{iK_2(t, x, y)} a_x a_y \right) dx dy \end{aligned}$$

Using Lemma 5 again, we obtain (21) with  $K_2$  replaced by  $-2iK_2$ , namely

$$\pm i[\mathbb{H}(t), \mathcal{N}] \leq \eta d\Gamma(1 - \Delta) + \frac{C_\varepsilon(\mathcal{N} + N^{\beta+\varepsilon})}{\eta(1+t)^3}$$

for all  $\eta > 0$ . This completes the proof.  $\square$

*Remark 8.* Assume that  $\Phi$  is the ground state of the quadratic Hamiltonian

$$\mathbb{H}_V(0) = d\Gamma(h(0) + V) + \frac{1}{2} \iint \left( K_2(0, x, y) a_x^* a_y^* + \overline{K_2(0, x, y)} a_x a_y \right) dx dy,$$

where  $V(x)$  is an appropriate trapping potential which ensures that  $h(0) + V \geq \eta > 0$  (in particular, this implies that  $h(0) + V \geq C_\eta^{-1}(1 - \Delta)$  for some constant  $C_\eta > 0$ ). By Lemma 5, we know that

$$\begin{aligned} \mathbb{H}_V(0) &\geq \frac{1}{2} d\Gamma(h(0) + V) + \frac{1}{2} \iint \left( K_2(0, x, y) a_x^* a_y^* + \overline{K_2(0, x, y)} a_x a_y \right) dx dy \\ &\geq -C\|(1 - \Delta_x)^{-1/2} K_2(0, \cdot, \cdot)\|_{L^2}^2 \geq -C_\varepsilon N^{\beta+\varepsilon}. \end{aligned}$$

In particular,  $\mathbb{H}_V(0)$  is bounded from below and it can be diagonalized by a Bogoliubov transformation. Moreover,  $\Phi$  is a quasi-free state and

$$\langle \Phi, \mathcal{N} \Phi \rangle \leq C\|(1 - \Delta_x)^{-1} K_2(0, \cdot, \cdot)\|_{L^2}^2 \leq C.$$

Here the first bound is a consequence of [19, Theorem 1 (ii)].

Since the ground state energy of  $\mathbb{H}_V(0)$  is always non-positive (see [16, Theorem 2.1 (i)]), we have

$$0 \geq \langle \Phi, \mathbb{H}_V(0) \Phi \rangle \geq \frac{1}{2} \langle \Phi, d\Gamma(h(0) + V) \Phi \rangle - C_\varepsilon N^{\beta+\varepsilon}.$$

Combining with  $h(0) + V \geq C_\eta^{-1}(1 - \Delta)$ , we obtain

$$\langle \Phi, d\Gamma(1 - \Delta) \Phi \rangle \leq C_\varepsilon N^{\beta+\varepsilon}, \quad \forall \varepsilon > 0.$$

This motivates the assumptions on  $\Phi(0)$  in Theorem 1.

We are ready to give

*Proof of Lemma 4.* From the Bogoliubov equation (5), we have

$$\partial_t \langle \Phi(t), \mathbb{H}(t) \Phi(t) \rangle = \langle \Phi(t), \partial_t \mathbb{H}(t) \Phi(t) \rangle$$

which implies that

$$\langle \Phi(t), \mathbb{H}(t) \Phi(t) \rangle - \langle \Phi(0), \mathbb{H}(0) \Phi(0) \rangle = \int_0^t \langle \Phi(s), \partial_s \mathbb{H}(s) \Phi(s) \rangle ds. \quad (26)$$

From the first bound in Lemma 7 with  $\eta = 1/2$ , we get

$$\begin{aligned} \pm \langle \Phi(t), (\mathbb{H}(t) + d\Gamma(\Delta)) \Phi(t) \rangle &\leq \frac{1}{2} \langle \Phi(t), d\Gamma(1 - \Delta) \Phi(t) \rangle \\ &\quad + C_\varepsilon \left( \langle \Phi(t), \mathcal{N} \Phi(t) \rangle + N^{\beta+\varepsilon} \right). \end{aligned}$$

This implies

$$\begin{aligned} \langle \Phi(0), \mathbb{H}(0) \Phi(0) \rangle &\leq \frac{3}{2} \langle \Phi(0), d\Gamma(1 - \Delta) \Phi(0) \rangle + \\ &\quad + C_\varepsilon \left( \langle \Phi(0), \mathcal{N} \Phi(0) \rangle + N^{\beta+\varepsilon} \right) \leq C_\varepsilon N^{\beta+\varepsilon} \end{aligned} \quad (27)$$

(where we have used the assumption on  $\Phi(0)$ ) and

$$\langle \Phi(t), \mathbb{H}(t) \Phi(t) \rangle \geq \frac{1}{2} \langle \Phi(t), d\Gamma(1 - \Delta) \Phi(t) \rangle - C_\varepsilon \left( \langle \Phi(t), \mathcal{N} \Phi(t) \rangle + N^{\beta+\varepsilon} \right). \quad (28)$$

Next, from the second bound in Lemma 7 with  $\eta = (1+t)^{-3/2}$  we have

$$\langle \Phi(t), \partial_t \mathbb{H}(t) \Phi(t) \rangle \leq C_\varepsilon \frac{\langle \Phi(t), d\Gamma(1 - \Delta) \Phi(t) \rangle + N^{\beta+\varepsilon}}{(1+t)^{3/2}}. \quad (29)$$

Inserting (27), (28) and (29) into (26), we obtain

$$\begin{aligned} \langle \Phi(t), d\Gamma(1 - \Delta) \Phi(t) \rangle &\leq C_\varepsilon \int_0^t \frac{\langle \Phi(s), d\Gamma(1 - \Delta) \Phi(s) \rangle}{(1+s)^{3/2}} ds \\ &\quad + C_\varepsilon \left( \langle \Phi(t), \mathcal{N} \Phi(t) \rangle + N^{\beta+\varepsilon} \right). \end{aligned} \quad (30)$$

Now instead of using the bound on  $\langle \Phi(t), \mathcal{N} \Phi(t) \rangle$  in Lemma 3, we present another argument which will be used again to deal with the many-body Schrödinger evolution in Section 4. From the Bogoliubov equation (5) and the third bound in Lemma 7 with  $\eta = (1+t)^{-3/2}$ , it follows that

$$\partial_t \langle \Phi(t), \mathcal{N} \Phi(t) \rangle = \langle \Phi(t), i[\mathbb{H}(t), \mathcal{N}] \Phi(t) \rangle \leq C_\varepsilon \frac{\langle \Phi(t), d\Gamma(1 - \Delta) \Phi(t) \rangle + N^{\beta+\varepsilon}}{(1+t)^{3/2}}.$$

Integrating over  $t$  and using the assumption on  $\Phi(0)$  we have

$$\langle \Phi(t), \mathcal{N} \Phi(t) \rangle \leq C_\varepsilon \int_0^t \frac{\langle \Phi(s), d\Gamma(1 - \Delta) \Phi(s) \rangle}{(1+s)^{3/2}} ds + C_\varepsilon N^{\beta+\varepsilon}.$$

Inserting the latter inequality into the right side of (30) we obtain

$$\langle \Phi(t), d\Gamma(1 - \Delta) \Phi(t) \rangle \leq C_\varepsilon \int_0^t \frac{\langle \Phi(s), d\Gamma(1 - \Delta) \Phi(s) \rangle}{(1+s)^{3/2}} ds + C_\varepsilon N^{\beta+\varepsilon}. \quad (31)$$

Now we define

$$f(t) := \int_0^t \frac{\langle \Phi(s), d\Gamma(1 - \Delta)\Phi(s) \rangle}{(1 + s)^{3/2}} ds + N^{\beta+\varepsilon}$$

and rewrite (31) as

$$\frac{d}{dt} \log(f(t)) = \frac{f'(t)}{f(t)} \leq \frac{C_\varepsilon}{(1 + t)^{3/2}}.$$

Integrating over  $t$  and using the fact that  $(1 + t)^{-3/2}$  is integrable on  $(0, \infty)$ , we get  $f(t) \leq C_\varepsilon N^{\beta+\varepsilon}$ . The desired result then follows from (31).  $\square$

#### 4. BOGOLIUBOV APPROXIMATION

Recall that any vector  $\Psi \in \mathfrak{H}^N$  can be decomposed uniquely as

$$\Psi = \sum_{n=0}^N u(t)^{\otimes(N-n)} \otimes_s \psi_n = \sum_{n=0}^N \frac{(a^*(u(t)))^{N-n}}{\sqrt{(N-n)!}} \psi_n$$

with  $\psi_n \in \mathfrak{H}_+(t)^n$ , see [16, Sec. 2.3]. This gives rise the unitary operator

$$\begin{aligned} U_N(t) : \mathfrak{H}^N &\rightarrow \mathcal{F}_+^{\leq N}(t) := \bigoplus_{n=0}^N \mathfrak{H}_+(t)^n \\ \Psi &\mapsto \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_N. \end{aligned}$$

Following [15], we reformulate the Schrödinger equation  $\Psi_N(t) = e^{-itH_N}\Psi(0)$  by introducing

$$\Phi_N(t) := U_N(t)\Psi_N(t)$$

which belongs to  $\mathcal{F}_+^{\leq N}(t)$  and satisfies the equation

$$\begin{cases} i\partial_t \Phi_N(t) = \tilde{H}_N(t)\Phi_N(t), \\ \Phi_N(0) = \mathbb{1}^{\leq N}\Phi(0). \end{cases} \quad (32)$$

Here  $\mathbb{1}^{\leq N}$  is the projection onto  $\mathcal{F}^{\leq N} = \mathbb{C} \oplus \mathfrak{H} \oplus \cdots \oplus \mathfrak{H}^N$  and

$$\tilde{H}_N(t) = \mathbb{1}^{\leq N} \left[ \mathbb{H}(t) + \frac{1}{2} \sum_{j=0}^4 (R_j + R_j^*) \right] \mathbb{1}^{\leq N}$$

with

$$R_0 = R_0^* = d\Gamma(Q(t)[w_N * |u(t)|^2 + \tilde{K}_1(t) - \mu_N(t)]Q(t)) \frac{1 - \mathcal{N}}{N - 1},$$

$$R_1 = -2 \frac{\mathcal{N}\sqrt{N - \mathcal{N}}}{N - 1} a(Q(t)[w_N * |u(t)|^2]u(t)),$$

$$R_2 = \iint K_2(t, x, y) a_x^* a_y^* dx dy \left( \frac{\sqrt{(N - \mathcal{N})(N - \mathcal{N} - 1)}}{N - 1} - 1 \right),$$

$$\begin{aligned} R_3 = \frac{\sqrt{N - \mathcal{N}}}{N - 1} \iiint (1 \otimes Q(t)w_N Q(t) \otimes Q(t))(x, y; x', y') \times \\ \times \overline{u(t, x)} a_y^* a_{x'} a_{y'} dx dy dx' dy', \end{aligned}$$

$$R_4 = R_4^* = \frac{1}{2(N - 1)} \iiint (Q(t) \otimes Q(t)w_N Q(t) \otimes Q(t))(x, y; x', y') \times$$

$$\times a_x^* a_y^* a_{x'} a_{y'} \, dx \, dy \, dx' \, dy'.$$

Here, in  $R_0$  and  $R_1$  we write  $w_N$  for the function  $w_N(x)$ , while in  $R_3$  and  $R_4$  we write  $w_N$  for the two-body multiplication operator  $w_N(x - y)$ .

In order to compare  $\Phi_N(t)$  with the Bogoliubov dynamics  $\Phi(t)$ , we need to bound all error terms  $R_j$ 's. In [18, Prop. 3], we proved that

$$(R_j + R_j^*) \mathbb{1}^{\leq N} (R_j + R_j^*) \leq C(N^{6\beta-2} + N^{3\beta-1})(1 + \mathcal{N})^4.$$

Unfortunately, this bound is only useful when  $\beta < 1/3$ . In the present paper, we will derive several improved estimates. Let us start with

**Lemma 9.** *We have the quadratic form estimates on  $\mathcal{F}_+^{\leq N}$ :*

$$\pm(R_j + R_j^*) \leq \eta \left( R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C(1 + \mathcal{N})}{\eta(1 + t)^3}, \quad \forall \eta > 0, \forall j = 0, 1, 2, 3,$$

and

$$0 \leq R_4 \leq CN^{3\beta-1}\mathcal{N}^2, \quad R_4 \leq CN^{\beta-1}d\Gamma(-\Delta)\mathcal{N}.$$

Here the constant  $C$  depends only on  $\|u(0, \cdot)\|_{W^{1,\ell}}$  (more precisely, on  $\kappa_0$  in the condition  $\|u(0, \cdot)\|_{W^{\ell,1}} \leq \kappa_0$ ).

*Proof.* Let us go term by term.

$j = 0$  Recall that

$$R_0 = d\Gamma \left( Q(t)[w_N * |u(t)|^2 + \tilde{K}_1(t) - \mu_N(t)]Q(t) \right) \frac{1 - \mathcal{N}}{N - 1}.$$

From the operator bounds in (16)-(17)-(18), we have

$$\pm d\Gamma \left( Q(t)[w_N * |u(t)|^2 + \tilde{K}_1(t) - \mu_N(t)]Q(t) \right) \leq \frac{CN}{(1 + t)^3}.$$

Since the left side of the latter inequality commutes with  $\mathcal{N}$ , we get

$$\begin{aligned} \pm R_0 &\leq \frac{CN^2}{N(1 + t)^3} \leq \eta \frac{\mathcal{N}^2}{N} + \eta^{-1} \frac{CN^2}{N(1 + t)^6} \\ &\leq \eta \frac{\mathcal{N}^2}{N} + \eta^{-1} \frac{CN}{(1 + t)^6}, \quad \forall \eta > 0. \end{aligned} \quad (33)$$

In the last inequality, we have used the fact that  $\mathcal{N} \leq N$  on  $\mathcal{F}_+^{\leq N}(t)$ .

$j = 1$  For every  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ , by the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\langle \Phi, R_1 \Phi \rangle| &= \frac{2}{N - 1} \left| \left\langle \Phi, \mathcal{N} \sqrt{N - \mathcal{N}} a \left( Q(t)[w_N * |u(t)|^2]u(t) \right) \Phi \right\rangle \right| \\ &\leq \frac{2}{N - 1} \|\mathcal{N} \sqrt{N - \mathcal{N}} \Phi\| \left\| a \left( Q(t)[w_N * |u(t)|^2]u(t) \right) \Phi \right\|. \end{aligned}$$

Now we use the elementary inequality  $a^*(v)a(v) \leq \|v\|_{L^2}^2 \mathcal{N}$  and

$$\left\| Q(t)[w_N * |u(t)|^2]u(t) \right\|_{L^2} \leq \|w_N * |u(t)|^2\|_{L^2} \|u(t)\|_{L^2} \leq \frac{C}{(1 + t)^3}.$$

Here the last estimate is (16). Thus

$$|\langle \Phi, R_1 \Phi \rangle| \leq \frac{C}{N(1 + t)^{3/2}} \langle \Phi, \mathcal{N}^2 (N - \mathcal{N}) \Phi \rangle^{1/2} \langle \Phi, \mathcal{N} \Phi \rangle^{1/2}$$

$$\leq \frac{\eta}{N} \langle \Phi, \mathcal{N}^2 \Phi \rangle + \frac{C}{\eta(1+t)^3} \langle \Phi, \mathcal{N} \Phi \rangle, \quad \forall \eta > 0. \quad (34)$$

In the last estimate we have used  $0 \leq N - \mathcal{N} \leq N$  on  $\mathcal{F}_+^{\leq N}(t)$ . Consequently,

$$\pm \langle \Phi, (R_1 + R_1^*) \Phi \rangle = \pm 2\Re \langle \Phi, R_1 \Phi \rangle \leq \frac{\eta}{N} \langle \Phi, \mathcal{N}^2 \Phi \rangle + \frac{C}{\eta(1+t)^3} \langle \Phi, \mathcal{N} \Phi \rangle$$

for all  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ . Therefore,

$$\pm (R_1 + R_1^*) \leq \eta \frac{\mathcal{N}^2}{N} + \eta^{-1} \frac{C\mathcal{N}}{(1+t)^3}, \quad \forall \eta > 0. \quad (35)$$

$j = 2$  For every  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ , we have

$$\begin{aligned} \langle \Phi, R_2 \Phi \rangle &= \iint K_2(t, x, y) \left\langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{(N - \mathcal{N})(N - \mathcal{N} - 1)}}{N - 1} - 1 \right) \Phi \right\rangle dx dy \\ &= \iint \tilde{K}_2(t, x, y) \left\langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{(N - \mathcal{N})(N - \mathcal{N} - 1)}}{N - 1} - 1 \right) \Phi \right\rangle dx dy. \end{aligned}$$

Here we can replace  $K_2(t) = Q(t) \otimes Q(t) \tilde{K}_2(t)$  by  $\tilde{K}_2(t)$ , namely replace the projection  $Q(t)$  by the identity, because  $\Phi$  belongs to the excited Fock space  $\mathcal{F}_+(t)$  (putting differently, this is because  $a(u)\Phi = 0$ ). By the Cauchy-Schwarz inequality, we can estimate

$$\begin{aligned} |\langle \Phi, R_2 \Phi \rangle| &\leq \iint |u(t, x)| w_N(x - y) |u(t, y)| \|a_x a_y \Phi\| \times \\ &\quad \times \left\| \left( \frac{\sqrt{(N - \mathcal{N})(N - \mathcal{N} - 1)}}{N - 1} - 1 \right) \Phi \right\| dx dy. \end{aligned}$$

Using  $0 \leq \mathcal{N} \leq N$  on  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ , it is straightforward to see that

$$\left\| \left( \frac{\sqrt{(N - \mathcal{N})(N - \mathcal{N} - 1)}}{N - 1} - 1 \right) \Phi \right\| \leq \frac{C}{\sqrt{N}} \langle \Phi, (1 + \mathcal{N}) \Phi \rangle^{1/2}. \quad (36)$$

Moreover, by the Cauchy-Schwarz inequality again,

$$\begin{aligned} &\iint |u(t, x)| w_N(x - y) |u(t, y)| \|a_x a_y \Phi\| dx dy \\ &\leq \left( \iint |u(t, x)|^2 w_N(x - y) |u(t, y)|^2 dx dy \right)^{1/2} \\ &\quad \times \left( \iint w_N(x - y) \|a_x a_y \Phi\|^2 dx dy \right)^{1/2} \leq \frac{C\sqrt{N}}{(1+t)^{3/2}} \langle \Phi, R_4 \Phi \rangle^{1/2}. \end{aligned}$$

In the last estimate we have used (17) and the definition of  $R_4$ . Thus

$$\begin{aligned} |\langle \Phi, R_2 \Phi \rangle| &\leq \frac{C}{(1+t)^{3/2}} \langle \Phi, R_4 \Phi \rangle^{1/2} \langle \Phi, (1 + \mathcal{N}) \Phi \rangle^{1/2} \\ &\leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C}{\eta(1+t)^3} \langle \Phi, (1 + \mathcal{N}) \Phi \rangle, \quad \forall \eta > 0. \end{aligned} \quad (37)$$

Consequently,

$$\pm (R_2 + R_2^*) \leq \eta R_4 + \frac{C(1 + \mathcal{N})}{\eta(1+t)^3}, \quad \forall \eta > 0. \quad (38)$$

$\boxed{j=3}$  For all  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ , by using the simplification involving the projection  $Q(t)$  as above and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
|\langle \Phi, R_3 \Phi \rangle| &= \frac{1}{N-1} \left| \iint w_N(x-y) \overline{u(t,x)} \langle \Phi, \sqrt{N-\mathcal{N}} a_y^* a_y a_x \Phi \rangle dx dy \right| \\
&\leq \frac{1}{N-1} \iint w_N(x-y) |u(t,x)| \cdot \|a_y \sqrt{N-\mathcal{N}} \Phi\| \cdot \|a_y a_x \Phi\| dx dy \\
&\leq \frac{2\|u(t,\cdot)\|_{L^\infty}}{N-1} \left( \iint w_N(x-y) \|a_x a_y \Phi\|^2 dx dy \right)^{1/2} \\
&\quad \times \left( \iint w_N(x-y) \|a_y \sqrt{N-\mathcal{N}} \Phi\|^2 dx dy \right)^{1/2} \\
&\leq \frac{C}{(1+t)^{3/2}} \langle \Phi, R_4 \Phi \rangle^{1/2} \langle \Phi, \mathcal{N} \Phi \rangle^{1/2} \\
&\leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C}{\eta(1+t)^3} \langle \Phi, \mathcal{N} \Phi \rangle, \quad \forall \eta > 0.
\end{aligned} \tag{39}$$

Thus we conclude that

$$\pm(R_3 + R_3^*) \leq \eta R_4 + \frac{C\mathcal{N}}{\eta(1+t)^3}, \quad \forall \eta > 0. \tag{40}$$

Collecting (33), (35), (38) and (40) gives us the first bound in Lemma 9.

$\boxed{j=4}$  The simple estimate  $0 \leq R_4 \leq N^{3\beta-1} \mathcal{N}^2$  follows from the uniform bound  $0 \leq w_N \leq CN^{3\beta}$ . Moreover, by Sobolev's inequality, we have

$$w_N(x-y) \leq C \|w_N\|_{L^{3/2}} (-\Delta_x) \leq CN^\beta (-\Delta_x - \Delta_y)$$

as quadratic form on  $\mathfrak{H}^2$  (see e.g. [21, Lemma 3.2] for a proof). Therefore,

$$R_4 \leq CN^{\beta-1} \iint (-\Delta_x - \Delta_y) a_x^* a_y^* a_x a_y dx dy \leq CN^{\beta-1} d\Gamma(-\Delta) \mathcal{N}.$$

This completes the proof of Lemma 9.  $\square$

Heuristically, the first estimate in Lemma 9 tells us that  $R_4$  is the main error term among all  $R_j$ 's. The simple bound  $0 \leq R_4 \leq N^{3\beta-1} \mathcal{N}^2$  can serve as a-priori estimate, but it is not sufficient when  $\beta > 1/3$ . On the other hand, in order to use the bound  $R_4 \leq CN^{\beta-1} d\Gamma(-\Delta) \mathcal{N}$  we need to control the kinetic energy  $\langle \Phi_N(t), d\Gamma(1-\Delta) \Phi_N(t) \rangle$ . We have

**Lemma 10.** *Under the assumptions in Theorem 1, we have*

$$\langle \Phi_N(t), d\Gamma(1-\Delta) \Phi_N(t) \rangle \leq C_\varepsilon N^{\beta+\varepsilon}, \quad \forall t > 0, \forall \varepsilon \in (0, 1-2\beta].$$

The proof of Lemma 10 is similar to that of Lemma 4. We will need

**Lemma 11.** *We have the quadratic form estimates on  $\mathcal{F}_+^{\leq N}$ :*

$$\begin{aligned}
\pm \partial_t(R_j + R_j^*) &\leq \eta \left( R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C(1+\mathcal{N})}{\eta(1+t)^3}, \\
\pm i[(R_j + R_j^*), \mathcal{N}] &\leq \eta \left( R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C(1+\mathcal{N})}{\eta(1+t)^3},
\end{aligned}$$

for all  $j = 0, 1, 2, 3, 4$  and  $\eta > 0$ . The constant  $C$  depends only on  $\|u(0, \cdot)\|_{W^{1,\ell}}$ .

*Proof.* First, we bound  $i[(R_j + R_j^*), \mathcal{N}]$ . If  $j = 0$  or  $j = 4$ , the commutator is 0. Moreover, we have

$$i[R_1, \mathcal{N}] = iR_1, \quad i[R_2, \mathcal{N}] = -2iR_2, \quad i[R_3, \mathcal{N}] = iR_3$$

because  $[a_x, \mathcal{N}] = a_x$ ,  $[a_x^* a_y^*, \mathcal{N}] = -2a_x^* a_y^*$  and  $[a_x^* a_y a_z, \mathcal{N}] = a_x^* a_y a_z$ , respectively. Thus the desired inequalities can be obtained in the same way as in Lemma 9 (more precisely, they follow from (34), (37) and (39)).

Next, we bound  $\partial_t(R_j + R_j^*)$  by proceeding as in the proof of Lemma 9. Let us explain term by term.

$j = 0$  From (16)-(17)-(18) and (22)-(23)-(24), we find that

$$\left\| \partial_t \left( Q(t) [w_N * |u(t)|^2 + \tilde{K}_1(t) - \mu_N(t)] Q(t) \right) \right\| \leq \frac{C}{(1+t)^3}.$$

Therefore, similarly to (33), we have

$$\begin{aligned} \pm \partial_t R_0 &= \pm d\Gamma \left( \partial_t \left( Q(t) [w_N * |u(t)|^2 + \tilde{K}_1(t) - \mu_N(t)] Q(t) \right) \right) \frac{1 - \mathcal{N}}{N - 1} \\ &\leq \frac{C\mathcal{N}^2}{N(1+t)^3} \leq \eta \frac{\mathcal{N}^2}{N} + \eta^{-1} \frac{C\mathcal{N}^2}{N(1+t)^6} \leq \eta \frac{\mathcal{N}^2}{N} + \eta^{-1} \frac{C\mathcal{N}}{(1+t)^6}, \quad \forall \eta > 0. \end{aligned} \quad (41)$$

$j = 1$  Using  $\|Q(t)\| \leq 1$ ,  $\|\partial_t Q(t)\| \leq C$ , (16) and (22), we have

$$\begin{aligned} &\left\| \partial_t \left( Q(t) [w_N * |u(t)|^2] u(t) \right) \right\|_{L^2} \\ &\leq C \| [w_N * |u(t)|^2] u(t) \|_{L^2} + \| [\partial_t (w_N * |u(t)|^2) u(t)] \|_{L^2} \leq \frac{C}{(1+t)^3}. \end{aligned}$$

Therefore, we can follow the proof of (35) and obtain

$$\pm \partial_t (R_1 + R_1^*) \leq \eta \frac{\mathcal{N}^2}{N} + \eta^{-1} \frac{C\mathcal{N}}{(1+t)^3}, \quad \forall \eta > 0. \quad (42)$$

$j = 2$  For every  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ , we have

$$\begin{aligned} \langle \Phi, \partial_t R_2 \Phi \rangle &= \iint \left[ (\partial_t Q(t) \otimes 1 + 1 \otimes \partial_t Q(t)) \tilde{K}_2(t, x, y) + \partial_t \tilde{K}_2(t, x, y) \right] \times \\ &\quad \times \left\langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{(N - \mathcal{N})(N - \mathcal{N} - 1)}}{N - 1} - 1 \right) \Phi \right\rangle dx dy. \end{aligned}$$

Here we have used the decomposition

$$\partial_t K_2(t) = \partial_t Q(t) \otimes Q(t) \tilde{K}_2(t) + Q(t) \otimes \partial_t Q(t) \tilde{K}_2(t) + Q(t) \otimes Q(t) \partial_t \tilde{K}_2(t)$$

and omitted the projection  $Q(t)$  using  $\Phi \in \mathcal{F}_+(t)$ . Similarly to (37), we have

$$\begin{aligned} &\left| \iint \partial_t \tilde{K}_2(t, x, y) \left\langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{(N - \mathcal{N})(N - \mathcal{N} - 1)}}{N - 1} - 1 \right) \Phi \right\rangle dx dy \right| \\ &\leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C}{\eta(1+t)^3} \langle \Phi, (1 + \mathcal{N}) \Phi \rangle, \quad \forall \eta > 0. \end{aligned}$$

The term involving  $(\partial_t Q(t) \otimes 1 + 1 \otimes \partial_t Q(t)) \tilde{K}_2(t, x, y)$  is bounded as

$$\left| \iint (\partial_t Q(t) \otimes 1 + 1 \otimes \partial_t Q(t)) \tilde{K}_2(t, x, y) \times \right.$$



$$\begin{aligned}
& \times \left\langle \Phi, a_x^* a_y^* \left( \frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N-1} - 1 \right) \Phi \right\rangle dx dy \Big| \\
& \leq \left( \iint \left| (\partial_t Q(t) \otimes 1 + 1 \otimes \partial_t Q(t)) \tilde{K}_2(t, x, y) \right|^2 dx dy \right)^{1/2} \times \\
& \quad \times \left( \iint \|a_x a_y \Phi\|^2 dx dy \right)^{1/2} \left\| \left( \frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N-1} - 1 \right) \Phi \right\| \\
& \leq \frac{C}{\sqrt{N}(1+t)^{3/2}} \langle \Phi, \mathcal{N}^2 \Phi \rangle^{1/2} \langle \Phi, (1+\mathcal{N}) \Phi \rangle^{1/2}.
\end{aligned}$$

Here we have used (12) and (36) in the last estimate. In summary,

$$|\langle \Phi, \partial_t R_2 \Phi \rangle| \leq \eta \left( \langle \Phi, R_4 \Phi \rangle + \frac{1}{N} \langle \Phi, \mathcal{N}^2 \Phi \rangle \right) + \frac{C}{\eta(1+t)^3} \langle \Phi, (1+\mathcal{N}) \Phi \rangle$$

for all  $\Phi \in \mathcal{F}_+^{\leq N}(t)$  and  $\eta > 0$ . Therefore,

$$\pm \partial_t (R_2 + R_2^*) \leq \eta \left( R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C(1+\mathcal{N})}{\eta(1+t)^3}, \quad \forall \eta > 0. \quad (43)$$

$j = 3$  For all  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ , we have

$$\begin{aligned}
\langle \Phi, \partial_t R_3 \Phi \rangle &= \frac{1}{N-1} \iiint \left[ (1 \otimes Q(t) w_N Q(t) \otimes Q(t))(x, y; x', y') \overline{\partial_t u(t, x)} \right. \\
&\quad \left. + \left( \partial_t (1 \otimes Q(t) w_N Q(t) \otimes Q(t)) \right)(x, y; x', y') \overline{u(t, x)} \right] \times \\
&\quad \times \langle \Phi, \sqrt{N-\mathcal{N}} a_y^* a_{x'} a_{y'} \Phi \rangle dx dy dx' dy'.
\end{aligned}$$

The term involving  $\partial_t u(t, x)$  can be estimated similarly to (39):

$$\begin{aligned}
& \frac{1}{N-1} \left| \iiint (1 \otimes Q(t) w_N Q(t) \otimes Q(t))(x, y; x', y') \overline{\partial_t u(t, x)} \times \right. \\
& \quad \left. \times \langle \Phi, \sqrt{N-\mathcal{N}} a_y^* a_{x'} a_{y'} \Phi \rangle dx dy dx' dy' \right| \\
& \leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C}{\eta(1+t)^3} \langle \Phi, \mathcal{N} \Phi \rangle, \quad \forall \eta > 0.
\end{aligned}$$

In the following, we will use the kernel estimate

$$|(\partial_t Q(t))(z; z')| = |\partial_t u(t, z) \overline{u(t, z')} + u(t, z) \overline{\partial_t u(t, z')}| \leq q(z) q(z') \quad (44)$$

where  $q(t, z) := |u(t, z)| + |\partial_t u(t, z)|$ . Recall that by Lemma 2,

$$\|q(t, \cdot)\|_{L^2} \leq C, \quad \|q(t, \cdot)\|_{L^\infty} \leq \frac{C}{(1+t)^{3/2}}.$$

Let us decompose  $\partial_t (1 \otimes Q(t) w_N Q(t) \otimes Q(t))$  into three terms. For the first term  $1 \otimes \partial_t Q(t) w_N Q(t) \otimes Q(t)$ , we can estimate

$$\begin{aligned}
& \frac{1}{N-1} \left| \iiint (1 \otimes \partial_t Q(t) w_N Q(t) \otimes Q(t))(x, y; x', y') \overline{u(t, x)} \times \right. \\
& \quad \left. \times \langle \Phi, \sqrt{N-\mathcal{N}} a_y^* a_{x'} a_{y'} \Phi \rangle dx dy dx' dy' \right|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N-1} \left| \iiint (\partial_t Q(t))(y; y') w_N(x-y') \delta(x-x') \overline{u(t, x)} \times \right. \\
&\quad \left. \times \langle \Phi, \sqrt{N-\mathcal{N}} a_y^* a_x a_{y'} \Phi \rangle \, dx \, dy \, dx' \, dy' \right| \\
&\leq \frac{1}{N-1} \iiint q(t, y) q(t, y') w_N(x-y') |u(t, x)| \times \\
&\quad \times \|a_y \sqrt{N-\mathcal{N}} \Phi\| \|a_x a_{y'} \Phi\| \, dx \, dy \, dy' \\
&\leq \frac{\|q(t, \cdot)\|_{L^\infty}}{N-1} \left( \int |q(t, y)|^2 \, dy \right)^{1/2} \left( \int \|a_y \sqrt{N-\mathcal{N}} \Phi\|^2 \, dy \right)^{1/2} \times \\
&\quad \times \left( \iint w_N(x-y') |u(t, x)|^2 \, dx \, dy' \right)^{1/2} \left( \iint w_N(x-y') \|a_x a_{y'} \Phi\|^2 \, dx \, dy' \right)^{1/2} \\
&\leq \frac{C}{(1+t)^{3/2}} \langle \Phi, R_4 \Phi \rangle^{1/2} \langle \Phi, \mathcal{N} \Phi \rangle^{1/2} \leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C \langle \Phi, \mathcal{N} \Phi \rangle}{\eta(1+t)^3}, \quad \forall \eta > 0.
\end{aligned}$$

For the second term  $1 \otimes Q(t) w_N \partial_t Q(t) \otimes Q(t)$ , we have

$$\begin{aligned}
&\frac{1}{N-1} \left| \iiint (1 \otimes Q(t) w_N \partial_t Q(t) \otimes Q(t))(x, y; x', y') \overline{u(t, x)} \times \right. \\
&\quad \left. \times \langle \Phi, \sqrt{N-\mathcal{N}} a_y^* a_{x'} a_{y'} \Phi \rangle \, dx \, dy \, dx' \, dy' \right| \\
&= \frac{1}{N-1} \left| \iiint w_N(x-y) (\partial_t Q(t))(x, x') \delta(y-y') \overline{u(t, x)} \times \right. \\
&\quad \left. \times \langle \Phi, \sqrt{N-\mathcal{N}} a_y^* a_{x'} a_{y'} \Phi \rangle \, dx \, dy \, dx' \, dy' \right| \\
&\leq \frac{1}{N-1} \iiint w_N(x-y) q(t, x) q(t, x') |u(t, x)| \times \\
&\quad \times \|a_y \sqrt{N-\mathcal{N}} \Phi\| \|a_{x'} a_{y'} \Phi\| \, dx \, dy \, dx' \\
&\leq \frac{1}{N-1} \|w_N\|_{L^1} \|q(t, \cdot)\|_{L^\infty} \|u(t, \cdot)\|_{L^\infty} \times \\
&\quad \times \left( \iint \|a_{x'} a_{y'} \Phi\|^2 \, dx' \, dy \right)^{1/2} \left( \iint |q(t, x')|^2 \|a_y \sqrt{N-\mathcal{N}} \Phi\|^2 \, dx' \, dy \right)^{1/2} \\
&\leq \frac{C}{\sqrt{N}(1+t)^3} \langle \Phi, \mathcal{N}^2 \Phi \rangle^{1/2} \langle \Phi, \mathcal{N} \Phi \rangle^{1/2} \leq \eta \frac{\langle \Phi, \mathcal{N}^2 \Phi \rangle}{N} + \frac{C \langle \Phi, \mathcal{N} \Phi \rangle}{\eta(1+t)^3}, \quad \forall \eta > 0.
\end{aligned}$$

The third term  $1 \otimes Q(t) w_N Q(t) \otimes \partial_t Q(t)$  is bounded similarly. Thus

$$|\langle \Phi, \partial_t R_3 \Phi \rangle| \leq \eta \left( \langle \Phi, R_4 \Phi \rangle + \frac{\langle \Phi, \mathcal{N}^2 \Phi \rangle}{N} \right) + \frac{C \langle \Phi, \mathcal{N} \Phi \rangle}{\eta(1+t)^3}$$

for all  $\Phi \in \mathcal{F}_+^{\leq N}(t)$  and  $\eta > 0$ . Consequently,

$$\pm \partial_t (R_3 + R_3^*) \leq \eta \left( R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C\mathcal{N}}{\eta(1+t)^3}, \quad \forall \eta > 0. \quad (45)$$

$j=4$  For all  $\Phi \in \mathcal{F}_+^{\leq N}(t)$ , we have

$$\begin{aligned}
\langle \Phi, \partial_t R_4 \Phi \rangle &= \frac{1}{2(N-1)} \Re \iiint \partial_t \left( Q(t) \otimes Q(t) w_N Q(t) \otimes Q(t) \right) (x, y; x', y') \\
&\quad \times \langle \Phi, a_x^* a_y^* a_{x'} a_{y'} \Phi \rangle \, dx \, dy \, dx' \, dy'.
\end{aligned}$$

Let us decompose  $\partial_t(Q(t) \otimes Q(t) w_N Q(t) \otimes Q(t))$  into four terms, and consider for example  $\partial_t Q(t) \otimes Q(t) w_N Q(t) \otimes Q(t)$ . Using (44) again, we have

$$\begin{aligned}
& \frac{1}{N-1} \left| \iiint \left( \partial_t Q(t) \otimes Q(t) w_N Q(t) \otimes Q(t) \right) (x, y; x', y') \times \right. \\
& \quad \left. \times \langle \Phi, a_x^* a_y^* a_{x'} a_{y'} \Phi \rangle \, dx \, dy \, dx' \, dy' \right| \\
&= \frac{1}{N-1} \left| \iiint (\partial_t Q(t))(x, x') w_N(x' - y) \delta(y - y') \times \right. \\
& \quad \left. \times \langle \Phi, a_x^* a_y^* a_{x'} a_{y'} \Phi \rangle \, dx \, dy \, dx' \, dy' \right| \\
&\leq \frac{1}{N-1} \iiint q(t, x) q(t, x') w_N(x' - y) \|a_x a_y \Phi\| \|a_{x'} a_{y'} \Phi\| \, dx \, dy \, dx' \\
&\leq \frac{\|q(t, \cdot)\|_{L^\infty}}{N-1} \left( \iiint w_N(x' - y) \|a_x a_y \Phi\|^2 \, dx \, dy \, dx' \right)^{1/2} \times \\
& \quad \times \left( \iiint w_N(x' - y) \|a_{x'} a_{y'} \Phi\|^2 |q(t, x)|^2 \, dx \, dy \, dx' \right)^{1/2} \\
&\leq \frac{C}{N^{1/2}(1+t)^{3/2}} \langle \Phi, \mathcal{N}^2 \Phi \rangle^{1/2} \langle \Phi, R_4 \Phi \rangle^{1/2} \\
&\leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C \langle \Phi, \mathcal{N}^2 \Phi \rangle}{\eta N (1+t)^3} \leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C \langle \Phi, \mathcal{N} \Phi \rangle}{\eta (1+t)^3}, \quad \forall \eta > 0.
\end{aligned}$$

By similar estimates, we find that

$$|\langle \Phi, \partial_t R_4 \Phi \rangle| \leq \eta \langle \Phi, R_4 \Phi \rangle + \frac{C \langle \Phi, \mathcal{N} \Phi \rangle}{\eta (1+t)^3}$$

for all  $\Phi \in \mathcal{F}_+^{\leq N}(t)$  and  $\eta > 0$ . Thus

$$\pm \partial_t R_4 \leq \eta R_4 + \frac{C \mathcal{N}}{\eta (1+t)^3}, \quad \forall \eta > 0. \quad (46)$$

This completes the proof.  $\square$

Now we are ready to provide

*Proof of Lemma 10.* We use the proof strategy of Lemma 4. Using the equation (32) we can write

$$\begin{aligned}
& \langle \Phi_N(t), \tilde{H}_N(t) \Phi_N(t) \rangle - \langle \Phi_N(0), \tilde{H}_N(0) \Phi_N(0) \rangle \\
&= \int_0^t \langle \Phi_N(s), \partial_s \tilde{H}_N(s) \Phi_N(s) \rangle \, ds \quad (47)
\end{aligned}$$

and

$$\langle \Phi_N(t), \mathcal{N} \Phi_N(t) \rangle - \langle \Phi_N(0), \mathcal{N} \Phi_N(0) \rangle = \int_0^t \langle \Phi_N(s), i[\tilde{H}_N(s), \mathcal{N}] \Phi_N(s) \rangle \, ds. \quad (48)$$

Let us estimate both sides of (47). Recall that

$$\tilde{H}_N(t) = \mathbf{1}^{\leq N} \left[ \mathbb{H}(t) + \frac{1}{2} \sum_{j=0}^4 (R_j + R_j^*) \right] \mathbf{1}^{\leq N}.$$

From Lemma 7 and Lemma 9, we have the form estimates on  $\mathcal{F}_+^{\leq N}(t)$ :

$$\begin{aligned} & \pm \mathbb{1}^{\leq N} \left( \tilde{H}_N(t) + d\Gamma(\Delta) - R_4 \right) \mathbb{1}^{\leq N} \\ &= \pm \mathbb{1}^{\leq N} \left( \mathbb{H}(t) + d\Gamma(\Delta) + \frac{1}{2} \sum_{j=0}^3 (R_j + R_j^*) \right) \mathbb{1}^{\leq N} \\ &\leq \eta \left( d\Gamma(1 - \Delta) + R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C_\varepsilon (N^{\beta+\varepsilon} + \mathcal{N})}{\eta(1+t)^3}. \end{aligned} \quad (49)$$

Similarly, from Lemma 7 and Lemma 11, we have

$$\pm \partial_t \tilde{H}_N(t) \leq \eta \left( d\Gamma(1 - \Delta) + R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C_\varepsilon (N^{\beta+\varepsilon} + \mathcal{N})}{\eta(1+t)^3}, \quad (50)$$

for all  $\eta > 0$ .

Applying (49) with  $\eta = 1/2$  and using  $\mathcal{N}^2/N \leq \mathcal{N} \leq d\Gamma(1 - \Delta)$  on  $\mathcal{F}_+^{\leq N}(t)$ , we find that

$$\begin{aligned} \langle \Phi_N(t), \tilde{H}_N(t) \Phi_N(t) \rangle &\geq \frac{1}{2} \langle \Phi_N(t), (d\Gamma(1 - \Delta) + R_4) \Phi_N(t) \rangle \\ &\quad - C_\varepsilon \left( N^{\beta+\varepsilon} + \langle \Phi_N(t), \mathcal{N} \Phi_N(t) \rangle \right), \end{aligned} \quad (51)$$

$$\langle \Phi_N(0), \tilde{H}_N(0) \Phi_N(0) \rangle \leq C \langle \Phi_N(0), (d\Gamma(1 - \Delta) + R_4) \Phi_N(0) \rangle + C_\varepsilon N^{\beta+\varepsilon}. \quad (52)$$

Using  $\Phi_N(0) = \mathbb{1}^{\leq N} \Phi(0)$ , we get

$$\langle \Phi_N(0), d\Gamma(1 - \Delta) \Phi_N(0) \rangle \leq \langle \Phi(0), d\Gamma(1 - \Delta) \Phi(0) \rangle \leq C_\varepsilon N^{\beta+\varepsilon}.$$

On the other hand, recall that  $R_4 \leq C N^{3\beta-1} \mathcal{N}^2$  by Lemma 9. Moreover, it is well-known that for every quasi-free state  $\Phi$  and  $s \in \mathbb{N}$ , we have

$$\langle \Phi, (1 + \mathcal{N})^s \Phi \rangle \leq C_s \langle \Phi, (1 + \mathcal{N}) \Phi \rangle^s \quad (53)$$

where the constant  $C_s$  depends only on  $s$  (see [18, Lemma 5] for a proof). Combining with the assumptions  $\langle \Phi(0), \mathcal{N} \Phi(0) \rangle \leq C_\varepsilon N^\varepsilon$  and  $\varepsilon \leq 1 - 2\beta$ , we obtain

$$\begin{aligned} \langle \Phi_N(0), R_4 \Phi_N(0) \rangle &\leq C N^{3\beta-1} \langle \Phi(0), \mathcal{N}^2 \Phi(0) \rangle \leq C N^{3\beta-1} \langle \Phi(0), \mathcal{N} \Phi(0) \rangle^2 \\ &\leq C_\varepsilon N^{3\beta-1} N^{2\varepsilon} \leq C_\varepsilon N^{\beta+\varepsilon}. \end{aligned}$$

Thus (52) reduces to

$$\langle \Phi_N(0), \tilde{H}_N(0) \Phi_N(0) \rangle \leq C_\varepsilon N^{\beta+\varepsilon}. \quad (54)$$

Next, we apply (50) with  $\eta = (1+t)^{-3/2}$  and use  $\mathcal{N}^2/N \leq \mathcal{N} \leq d\Gamma(1 - \Delta)$  on  $\mathcal{F}_+^{\leq N}(t)$ . This gives

$$\langle \Phi_N(t), \partial_t \tilde{H}_N(t) \Phi_N(t) \rangle \leq C_\varepsilon \frac{\langle \Phi_N(t), (d\Gamma(1 - \Delta) + R_4) \Phi_N(t) \rangle + N^{\beta+\varepsilon}}{(1+t)^{3/2}}. \quad (55)$$

Inserting (51), (54) and (55) into (47) we obtain

$$\langle \Phi_N(t), (d\Gamma(1 - \Delta) + R_4) \Phi_N(t) \rangle \leq C_\varepsilon \int_0^t \frac{\langle \Phi_N(s), (d\Gamma(1 - \Delta) + R_4) \Phi_N(s) \rangle}{(1+s)^{3/2}} ds$$

$$+ C_\varepsilon \left( N^{\beta+\varepsilon} + \langle \Phi_N(t), \mathcal{N}\Phi_N(t) \rangle \right). \quad (56)$$

Now we consider (48). By using  $\Phi(0) = \mathbf{1}^{\leq N} \Phi(0)$  and the assumption on  $\Phi(0)$ , we have

$$\langle \Phi_N(0), \mathcal{N}\Phi_N(0) \rangle \leq \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \leq C_\varepsilon N^{\beta+\varepsilon}.$$

Moreover, from Lemma 7 and Lemma 11, we have

$$\pm i[\tilde{H}_N(t), \mathcal{N}] \leq \eta \left( d\Gamma(1 - \Delta) + R_4 + \frac{\mathcal{N}^2}{N} \right) + \frac{C_\varepsilon(N^{\beta+\varepsilon} + \mathcal{N})}{\eta(1+t)^3}, \quad \forall \eta > 0.$$

We can choose  $\eta = (1+t)^{-3/2}$  and use  $\mathcal{N}^2/N \leq \mathcal{N} \leq d\Gamma(1 - \Delta)$  on  $\mathcal{F}_+^{\leq N}(t)$  to obtain

$$\pm i[\tilde{H}_N(t), \mathcal{N}] \leq C_\varepsilon \frac{d\Gamma(1 - \Delta) + R_4 + N^{\beta+\varepsilon}}{(1+t)^{3/2}}.$$

Inserting the latter estimate into the right side of (48), we get

$$\langle \Phi(t), \mathcal{N}\Phi(t) \rangle \leq C_\varepsilon \int_0^t \frac{\langle \Phi_N(s), (d\Gamma(1 - \Delta) + R_4)\Phi_N(s) \rangle}{(1+s)^{3/2}} ds + C_\varepsilon N^{\beta+\varepsilon}. \quad (57)$$

Finally, we substitute (57) into the right side of (56) and find that

$$\begin{aligned} & \langle \Phi_N(t), (d\Gamma(1 - \Delta) + R_4)\Phi_N(t) \rangle \\ & \leq C_\varepsilon \int_0^t \frac{\langle \Phi_N(s), (d\Gamma(1 - \Delta) + R_4)\Phi_N(s) \rangle}{(1+s)^{3/2}} ds + C_\varepsilon N^{\beta+\varepsilon}. \end{aligned} \quad (58)$$

This bound is similar to (30) and we can argue as in the proof of Lemma 4 to conclude that

$$\langle \Phi_N(t), (d\Gamma(1 - \Delta) + R_4)\Phi_N(t) \rangle \leq C_\varepsilon N^{\beta+\varepsilon}.$$

Since  $R_4 \geq 0$ , the desired kinetic estimate follows.  $\square$

## 5. PROOF OF THE MAIN THEOREM

*Proof of Theorem 1.* It suffices to consider the case when  $N$  is large and  $\varepsilon$  is small (indeed, if the desired bound holds for some  $\varepsilon > 0$ , then it also holds for any  $\varepsilon' > \varepsilon$  because  $(1+t)^{1+\varepsilon} N^{(2\beta+\varepsilon-1)/2} \leq (1+t)^{1+\varepsilon'} N^{(2\beta+\varepsilon'-1)/2}$ ). In particular, we will assume  $\varepsilon < \min\{1/2, 1-2\beta\}$ .

Since  $U_N(t)$  is a unitary operator from  $\mathfrak{H}^N$  to  $\mathcal{F}_+^{\leq N}(t) \subset \mathcal{F}(\mathfrak{H})$ , we have

$$\begin{aligned} \|\Psi_N(t) - U_N(t)^* \mathbf{1}^{\leq N} \Phi(t)\|_{\mathfrak{H}^N} &= \|U_N(t)\Psi_N(t) - \mathbf{1}^{\leq N} \Phi(t)\| \\ &= \|\mathbf{1}^{\leq N}(\Phi_N(t) - \Phi(t))\| \leq \|\Phi_N(t) - \Phi(t)\|. \end{aligned} \quad (59)$$

Using the equations (5) and (32), we can compute

$$\begin{aligned} \partial_t \|\Phi_N(t) - \Phi(t)\|^2 &= -2\Re \partial_t \langle \Phi_N(t), \Phi(t) \rangle \\ &= -2\Re \left( \langle \partial_t \Phi_N(t), \Phi(t) \rangle + \langle \Phi_N(t), \partial_t \Phi(t) \rangle \right) \\ &= -2\Re \left( \langle -i\tilde{H}_N(t)\Phi_N(t), \Phi(t) \rangle + \langle \Phi_N(t), -i\mathbb{H}(t)\Phi(t) \rangle \right) \\ &= 2\Re \langle i\Phi_N(t), (\tilde{H}_N(t) - \mathbb{H}(t))\Phi(t) \rangle. \end{aligned}$$

Since

$$\tilde{H}_N(t) = \mathbb{1}^{\leq N} \left[ \mathbb{H}(t) + \frac{1}{2} \sum_{j=0}^4 (R_j + R_j^*) \right] \mathbb{1}^{\leq N}$$

and  $\Phi_N \in \mathcal{F}_+^{\leq N}(t)$ , we can define  $\mathbb{1}^{>N} := \mathbb{1} - \mathbb{1}^{\leq N}$  and write

$$\begin{aligned} \partial_t \|\Phi_N(t) - \Phi(t)\|^2 &= \sum_{j=0}^4 \Re \langle i\Phi_N(t), (R_j + R_j^*) \mathbb{1}^{\leq N} \Phi(t) \rangle \\ &\quad - 2\Re \langle i\Phi_N(t), \mathbb{H} \mathbb{1}^{>N} \Phi(t) \rangle. \end{aligned} \quad (60)$$

**Step 1.** Let us consider the last term of (60). Since  $\Phi_N(t) \in \mathcal{F}_+^{\leq N}(t)$  and  $\mathbb{1}^{\leq N} d\Gamma(h) \mathbb{1}^{>N} = 0$ , we have

$$\langle \Phi_N(t), \mathbb{H} \mathbb{1}^{>N} \Phi(t) \rangle = \langle \Phi_N(t), (\mathbb{H} - d\Gamma(h)) \mathbb{1}^{>N} \Phi(t) \rangle.$$

Recall that by (10),

$$\pm (\mathbb{H} - d\Gamma(h)) \leq C(\mathcal{N} + N^{3\beta}). \quad (61)$$

Here  $C$  is a general constant depending only on  $\|u(0, \cdot)\|_{W^{\ell,1}}$  (more precisely, on  $\kappa_0$  in the condition  $\|u(0, \cdot)\|_{W^{\ell,1}} \leq \kappa_0$ ).

We will use the general fact that if  $A$  and  $B$  are quadratic forms satisfying  $\pm B \leq A$ , then for all vectors  $f, g$  we have the Cauchy-Schwarz type inequality

$$\begin{aligned} |\langle f, Bg \rangle| &\leq |\langle f, (A+B)g \rangle| + |\langle f, Ag \rangle| \\ &\leq \langle f, (A+B)f \rangle^{1/2} \langle g, (A+B)g \rangle^{1/2} + \langle f, Af \rangle^{1/2} \langle g, Ag \rangle^{1/2} \\ &\leq 3\langle f, Af \rangle^{1/2} \langle g, Ag \rangle^{1/2}. \end{aligned} \quad (62)$$

From (61) and (62), we obtain

$$\begin{aligned} |\langle \Phi_N(t), \mathbb{H} \mathbb{1}^{>N} \Phi(t) \rangle| &= |\langle \Phi_N(t), (\mathbb{H} - d\Gamma(h)) \mathbb{1}^{>N} \Phi(t) \rangle| \\ &\leq C \left\langle \Phi_N(t), (\mathcal{N} + N^{3\beta}) \Phi_N(t) \right\rangle^{1/2} \left\langle \mathbb{1}^{>N} \Phi(t), (\mathcal{N} + N^{3\beta}) \mathbb{1}^{>N} \Phi(t) \right\rangle^{1/2} \\ &\leq C(N + N^{3\beta}) \left\langle \Phi(t), \mathcal{N}^s N^{-s} \Phi(t) \right\rangle^{1/2}, \quad \forall s \geq 1. \end{aligned} \quad (63)$$

Here, in the last inequality, we have used  $\mathcal{N} \leq N$  on  $\mathcal{F}_+^{\leq N}(t)$  and

$$\mathbb{1}^{>N} (\mathcal{N} + N^{3\beta}) \leq (N + N^{3\beta}) \mathcal{N}^s N^{-s}, \quad \forall s \geq 1.$$

Now we use the moment estimate (53), the bound on  $\langle \Phi(t), \mathcal{N} \Phi(t) \rangle$  in Lemma 3 and the assumption  $\langle \Phi(0), \mathcal{N} \Phi(0) \rangle \leq C_\varepsilon N^\varepsilon$ . All this gives

$$\langle \Phi(t), (1 + \mathcal{N})^s \Phi(t) \rangle \leq C_s \langle \Phi(t), (1 + \mathcal{N}) \Phi(t) \rangle^s \leq C_{\varepsilon,s} N^{2s\varepsilon} [\log(2+t)]^{2s}. \quad (64)$$

Hence, (63) reduces to

$$|\langle \Phi_N(t), \mathbb{H} \mathbb{1}^{>N} \Phi(t) \rangle| \leq C_{\varepsilon,s} (N + N^{3\beta}) N^{s(\varepsilon-1/2)} [\log(2+t)]^s$$

for all  $s \geq 1$ . Since  $\varepsilon - 1/2 < 0$ , we can choose  $s = s(\varepsilon)$  sufficiently large (e.g.  $s \geq (2 + 3\beta)/(1/2 - \varepsilon)$ ) to obtain

$$|\langle \Phi_N(t), \mathbb{H} \mathbb{1}^{>N} \Phi(t) \rangle| \leq C_\varepsilon N^{-1} (1+t)^\varepsilon. \quad (65)$$

Here we have bound  $[\log(2+t)]^{s_\varepsilon}$  by  $C_\varepsilon(1+t)^\varepsilon$  for simplicity.

**Step 2.** Now we turn to the first term on the right side of (60). Recall that by Lemma 9, we have the quadratic form estimates on  $\mathcal{F}_+^{\leq N}(t)$ :

$$\pm(R_j + R_j^*) \leq 2(1+\eta)R_4 + \eta \frac{\mathcal{N}^2}{N} + \frac{C(1+\mathcal{N})}{\eta(1+t)^3} \quad (66)$$

for all  $j = 0, 1, 2, 3, 4$  and  $\eta > 0$  (the bound for  $j = 4$  does not follow from Lemma 9 but it is trivial).

Since we do not have a good control on  $\langle \Phi_N(t), \mathcal{N}\Phi_N(t) \rangle$ , we need to introduce a cut-off before applying (66). Note that for every  $4 < M < N-2$ ,

$$\mathbf{1}^{\leq M}(R_j + R_j^*)\mathbf{1}^{>M+2} = 0 \quad \text{and} \quad \mathbf{1}^{>M}(R_j + R_j^*)\mathbf{1}^{\leq M-2} = 0$$

because there are at most 2 creation or annihilation operators in the expressions of  $R_j$ 's. Therefore, we can write

$$\begin{aligned} \langle \Phi_N(t), (R_j + R_j^*)\mathbf{1}^{\leq N}\Phi(t) \rangle &= \langle \mathbf{1}^{\leq M}\Phi_N(t), (R_j + R_j^*)\mathbf{1}^{\leq M+2}\Phi(t) \rangle \\ &\quad + \langle \mathbf{1}^{>M}\Phi_N(t), (R_j + R_j^*)\mathbf{1}^{\leq N}\mathbf{1}^{>M-2}\Phi(t) \rangle \end{aligned}$$

and then apply (66) and (62) to each term on the right side. This gives

$$|\langle \Phi_N(t), (R_j + R_j^*)\mathbf{1}^{\leq N}\Phi(t) \rangle| \leq C(E_1 + E_2) \quad (67)$$

where

$$\begin{aligned} E_1 &= \inf_{\eta>0} \left\langle \mathbf{1}^{\leq M}\Phi_N(t), \left( (1+\eta)R_4 + \eta \frac{\mathcal{N}^2}{N} + \frac{1+\mathcal{N}}{\eta(1+t)^3} \right) \mathbf{1}^{\leq M}\Phi_N(t) \right\rangle^{1/2} \\ &\quad \times \left\langle \mathbf{1}^{\leq M+2}\Phi(t), \left( (1+\eta)R_4 + \eta \frac{\mathcal{N}^2}{N} + \frac{1+\mathcal{N}}{\eta(1+t)^3} \right) \mathbf{1}^{\leq M+2}\Phi(t) \right\rangle^{1/2}, \\ E_2 &= \inf_{\eta>0} \left\langle \mathbf{1}^{>M}\Phi_N(t), \left( (1+\eta)R_4 + \eta \frac{\mathcal{N}^2}{N} + \frac{1+\mathcal{N}}{\eta(1+t)^3} \right) \mathbf{1}^{>M}\Phi_N(t) \right\rangle^{1/2} \\ &\quad \times \left\langle \mathbf{1}^{>M-2}\Phi(t), \left( (1+\eta)R_4 + \eta \frac{\mathcal{N}^2}{N} + \frac{1+\mathcal{N}}{\eta(1+t)^3} \right) \mathbf{1}^{>M-2}\Phi(t) \right\rangle^{1/2}. \end{aligned}$$

To bound  $E_1$ , we use the last estimate in Lemma 9 and  $\mathbf{1}^{\leq M}\mathcal{N} \leq M$ :

$$\mathbf{1}^{\leq M}R_4 \leq CN^{\beta-1}\mathbf{1}^{\leq M}\mathcal{N}d\Gamma(-\Delta) \leq CN^{\beta-1}Md\Gamma(-\Delta).$$

Moreover, recall that we have the kinetic estimate in Lemma 10:

$$\langle \Phi_N(t), d\Gamma(1-\Delta)\Phi_N(t) \rangle \leq C_\varepsilon N^{\beta+\varepsilon}, \quad \forall \varepsilon \in (0, 1-2\beta],$$

where the constant  $C_\varepsilon$  is independent of  $N$  and  $t$ . Therefore,

$$\begin{aligned} &\left\langle \mathbf{1}^{\leq M}\Phi_N(t), \left( (1+\eta)R_4 + \eta \frac{\mathcal{N}^2}{N} + \frac{1+\mathcal{N}}{\eta(1+t)^3} \right) \mathbf{1}^{\leq M}\Phi_N(t) \right\rangle \\ &\leq C_\varepsilon \left( (1+\eta)N^{\beta-1}MN^{\beta+\varepsilon} + \eta M^2N^{-1} + \frac{M}{\eta(1+t)^3} \right). \end{aligned}$$

Similarly, we have the same bound with  $\mathbf{1}^{\leq M}\Phi_N(t)$  replaced by  $\mathbf{1}^{\leq M+2}\Phi(t)$  (the kinetic estimate for  $\Phi(t)$  is provided in Lemma 4). Then by optimizing over  $\eta > 0$  we find that

$$E_1 \leq C_\varepsilon \left( MN^{(2\beta+\varepsilon-1)/2} + M^{3/2}N^{-1/2} \right).$$

Next, we bound  $E_2$  using the argument in Step 1. To be precise, let us choose  $\eta = 1$  in the variational formula of  $E_2$  for simplicity and then use  $R_4 \leq CN^{3\beta-1}\mathcal{N}^2$  (see Lemma 9). We obtain the rough bound

$$E_2 \leq CN^{3\beta} \langle \mathbb{1}^{>M} \Phi_N(t), (\mathcal{N} + 1)^2 \mathbb{1}^{>M} \Phi_N(t) \rangle^{1/2} \\ \times \langle \mathbb{1}^{>M-2} \Phi(t), (\mathcal{N} + 1)^2 \mathbb{1}^{>M-2} \Phi(t) \rangle^{1/2}.$$

Now for the first term we use  $\mathbb{1}^{\leq N}(\mathcal{N} + 1) \leq N + 1$  (recall that  $\Phi_N(t) \in \mathcal{F}_+^{\leq N}(t)$ ) and get

$$\langle \mathbb{1}^{>M} \Phi_N(t), (\mathcal{N} + 1)^2 \mathbb{1}^{>M} \Phi_N(t) \rangle \leq (N + 1)^2.$$

For the second term, we use  $\mathbb{1}^{>M-2}(\mathcal{N} + 1)^2 \leq (\mathcal{N} + 1)^s(M - 1)^{2-s}$  with  $s \geq 2$  and then use the moment estimate (64). We find that

$$\langle \mathbb{1}^{>M-2} \Phi(t), (\mathcal{N} + 1)^2 \mathbb{1}^{>M-2} \Phi(t) \rangle \leq (M - 1)^{2-s} \langle \Phi(t), (\mathcal{N} + 1)^s \Phi(t) \rangle \\ \leq C_{\varepsilon, s} (M - 1)^{2-s} N^{2s\varepsilon} [\log(2 + t)]^{2s}.$$

All this yields

$$E_2 \leq C_{\varepsilon, s} N^{3\beta+1} M^{1-s/2} N^{s\varepsilon} [\log(2 + t)]^s.$$

In summary, from (67) it follows that

$$|\langle \Phi_N(t), (R_j + R_j^*) \mathbb{1}^{\leq N} \Phi(t) \rangle| \leq C_\varepsilon \left( MN^{(2\beta+\varepsilon-1)/2} + M^{3/2} N^{-1/2} \right) \\ + C_{\varepsilon, s} N^{3\beta+1} M^{1-s/2} N^{s\varepsilon} [\log(2 + t)]^s$$

for all  $4 < M < N - 2$  and  $s \geq 2$ . We can choose  $M = N^{3\varepsilon}$  and  $s = s(\varepsilon)$  sufficiently large (e.g.  $s \geq 6(1 + \beta + \varepsilon)/\varepsilon$ ) to obtain

$$|\langle \Phi_N(t), (R_j + R_j^*) \mathbb{1}^{\leq N} \Phi(t) \rangle| \leq C_\varepsilon \left( N^{(2\beta+9\varepsilon-1)/2} + N^{-1}(1 + t)^\varepsilon \right). \quad (68)$$

**Step 3.** From (60), (65) and (68), we find that

$$\partial_t \|\Phi_N(t) - \Phi(t)\|^2 \leq C_\varepsilon \left( N^{(2\beta+9\varepsilon-1)/2} + N^{-1}(1 + t)^\varepsilon \right).$$

Integrating over  $t$  and using

$$\|\Phi_N(0) - \Phi(0)\|^2 = \langle \Phi(0), \mathbb{1}^{>N} \Phi(0) \rangle \leq N^{-1} \langle \Phi(0), \mathcal{N} \Phi(0) \rangle \leq C_\varepsilon N^{\varepsilon-1}.$$

we obtain

$$\|\Phi_N(t) - \Phi(t)\|^2 \leq C_\varepsilon N^{\varepsilon-1} + C_\varepsilon \left( t N^{(2\beta+9\varepsilon-1)/2} + N^{-1}(1 + t)^{1+\varepsilon} \right) \\ \leq C_\varepsilon (1 + t)^{1+\varepsilon} N^{(2\beta+9\varepsilon-1)/2}.$$

Finally, from (59) we conclude that

$$\|\Psi_N(t) - U_N(t)^* \mathbb{1}^{\leq N} \Phi(t)\|_{\mathfrak{H}_N}^2 \leq \|\Phi_N(t) - \Phi(t)\|^2 \\ \leq C_\varepsilon (1 + t)^{1+\varepsilon} N^{(2\beta+9\varepsilon-1)/2} \quad (69)$$

for all  $0 < \varepsilon < \min\{1/2, 1 - 2\beta\}$ . In the latter estimate we can thus replace  $\varepsilon$  by  $\varepsilon/9$  and obtain

$$\|\Psi_N(t) - U_N(t)^* \mathbb{1}^{\leq N} \Phi(t)\|_{\mathfrak{H}_N}^2 \leq C_\varepsilon (1 + t)^{1+\varepsilon} N^{(2\beta+\varepsilon-1)/2}$$

for all  $0 < \varepsilon < \min\{1/2, 1 - 2\beta\}$  (with the constant  $C_\varepsilon$  adjusted appropriately). As we have explained, this estimate holds for all  $\varepsilon > 0$  because



$(1+t)^{1+\varepsilon} N^{(2\beta+\varepsilon-1)/2} \leq (1+t)^{1+\varepsilon'} N^{(2\beta+\varepsilon'-1)/2}$  when  $\varepsilon' \geq \varepsilon$ . This ends the proof of Theorem 1.  $\square$

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